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Generalized Kramers–Wannier duality for spin systems with non-commutative symmetry

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Abstract

In 1941 H Kramers and G Wannier discovered a special symmetry which relates low-temperature and high-temperature phases in the planar Ising model. The corresponding transformation, the Kramers–Wannier transform, is a special non-local substitution in the partition function. The existence of such transformations is a general property of lattice spin systems. Generalization of the KW transform to spin systems with non-Abelian symmetry is essential for many problems in statistical physics and field theory. This problem is very difficult and cannot be carried out by classical methods (like Fourier transform in the commutative case). We present new results which solve this problem for finite non-Abelian groups.

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Introduction

In the classical paper of Kramers and Wannier [1] a special symmetry was discovered, which relates low-temperature and high-temperature phases in the planar Ising model. The corresponding transformation, the Kramers–Wannier (KW) transform, is a special nonlocal substitution of a variable in the partition function. This substitution transforms the partition function W defined by the initial 'spin' variables taking values in Z_2 and determined on the vertices of the original lattice L to the partition function $\tilde{W}$ determined on the dual lattice L^* spin variables taking values in Z_2.

Furthermore, we will use the following transformation of the Boltzmann factor

$$\beta \to \beta^* = \operatorname{arth} \mathrm{e}^{-2\beta} \qquad \beta = (kT)^{-1} \tag{0.1}$$

to get the correct form of the dual partition function $\tilde{W}$.

The existence of such transformations is a general property of lattice spin systems that possess a discrete (and not only discrete) group of symmetry. The KW transform allows the determination, for many physically important systems, of the point of phase transition in the cases when the explicit analytical form of a partition function is unknown.

Generalizations of the KW transform in spin systems with different symmetry groups is essential for many problems in statistical physics and field theory. In fact, it is very important to carry out KW transforms for four-dimensional gauge theories in which the corresponding phases are free quarks and quark confinement. In this case we need to construct KW transforms for non-Abelian groups.

The KW transform for systems with a commutative symmetry group K, particularly Z_n and Z (like the Ising Z_2-model), can be carried out by general methods. In this case the KW transform is a Fourier transform from a spin system on the lattice L to the spin system on the dual lattice $\tilde{L}$ with spin variables taking values in group $\hat{K}$, the group of characters of K. This result was obtained by a number of authors, see [2–4] and references therein. From the mathematical point of view this result is a generalization of the classical Poisson summation formula for the group Z.

In this paper we present new results which solve this problem for finite non-commutative groups. Our method was inspired by the recent achievements in the theory of multivalued groups [5].

The efficacy of our approach will be illustrated by examples of KW transforms for the icosahedron I_5 and dihedral groups D_n. These examples are also interesting for physical applications, for example, to search out the line of phase transitions in quasicrystals with icosahedral symmetry or discotic liquid crystals with the symmetry D_n.

The main result of our paper is the definition of the generalized KW transform, based on the mapping of the group algebra $C(G)$ to the space of complex-valued functions on G. The construction of this transformation clarifies its real meaning and offers far-reaching generalization [2, 6, 7].

The layout of the paper is as follows. In section 1 we recall, following [2], the construction of the KW transform for Abelian groups. In section 2 we introduce some relevant algebraic notions such as the group algebra $C(G)$ and the space of regular functions $C[G]$. We also construct the canonical pairing of $C(G)$ with $C[G]$. In section 3 we describe orbits of the adjoint representation and the regular representation of the group G. In section 4 we carry out the generalized KW transform for finite groups. In section 5 we apply our general results to special cases of subgroups of the group $SO(3)$, including I_5 and D_n. In the conclusion we discuss some applications of these results, in particular some connections with quantum groups.

1. KW duality for Abelian systems

Let us recall the construction of KW duality for commutative groups. We shall follow [2]. Let us consider a planar square lattice L with unit edge. Let $x = \{x_\mu\} = \{x_1, x_2\}$ (where x_1 and x_2 are integers) represent a vertex, and $e_\mu^\alpha = \left\{e_\mu^1, e_\mu^2\right\} = \delta_\mu^\alpha$ basis vectors of L. We will often use the notation $x + \hat{\alpha} \equiv \left\{x_\mu + e_\mu^\alpha\right\}$. A double index x, α is convenient for denoting the edge in the lattice which connects the vertices x and $x + \hat{\alpha}$. In what follows we shall also need the dual lattice, $\tilde{L}$ whose vertices are at the centres of the faces of the original lattice L. We denote the coordinates of a vertice of $\tilde{L}$ by $\tilde{x}$:

$$\tilde{x} = \left\{x_\mu + \tfrac{1}{2}e_\mu^1 + \tfrac{1}{2}e_\mu^2\right\}.$$

We define spin variables s_x on vertices of L, these take values in some manifold M, which we call the spin space. We confine ourselves to the case of a finite set M.

The simplest Hamiltonian of such a spin system involves only interactions of nearest neighbours

$$\mathcal{H} = \sum_{x,\alpha} H(s_x, s_{x+\hat{\alpha}}) \tag{1.1}$$

where the Hamiltonian $H(s, s')$ is a real function of a pair of points from M, with the properties

$$H(s, s') = H(s', s) \tag{1.2a}$$

$$H(s, s') \geqslant 0 \quad \text{for arbitrary } s \qquad s' \in M \qquad H(s, s) = 0. \tag{1.2b}$$

The Hamiltonian prescribes a structure similar on M to a metric structure (which in the general case is not metric, since we nowhere require that the triangle inequality holds), which we shall call the H structure.

Of particular interest are examples in which the manifold M is a homogeneous space, i.e., there exists a group G of transformations of M which preserves the H structure: $H(gs, gs') = H(s, s')$ for arbitrary $s, s' \in M$. In this case the spin system has global symmetry with group G.

Important special cases are systems on groups. For these the spin manifold coincides with a group G: $s_i = g_i \in G$, and the Hamiltonian is invariant under left and right translations:

$$H(hg, hg') = H(gh, g'h) = H(g, g') \qquad \text{for arbitrary } h \in G. \tag{1.3}$$

The general H function of the system on the group can therefore be put in the form

$$H(g_1, g_2) = H\left(g_1 g_2^{-1}\right) = \sum_p h(p)\chi_p\left(g_1 g_2^{-1}\right) \tag{1.4}$$

where $\chi_p(g)$ are the characters of the pth irreducible representations of the group G, and the constants $h(p)$ are chosen so that H has the properties (1.2) and are otherwise arbitrary.

The partition function of the general spin system with the Hamiltonian (1.1) is

$$Z = \sum_{s_x \in M} \prod_{x,\alpha} W(s_x, s_{x+\alpha}) \tag{1.5}$$

where

$$W(s, s') = \exp\{-H(s, s')\}. \tag{1.6}$$

According to equation (1.2) the function W has the properties

$$W(s, s') = W(s', s) \qquad 0 \leqslant W(s, s') \leqslant 1 \qquad W(s, s) = 1. \tag{1.7}$$

For the system on a group we have also

$$W(g_1, g_2) = W\left(g_1 g_2^{-1}\right) \qquad W(g^{-1}) = W(g). \tag{1.8}$$

For a spin system on a group G the sum over states (1.5) can be put in the following equivalent form:

$$Z = \sum_{g_{x,\alpha} \in G} \prod_{x,\alpha} W(g_{x,\alpha}) \prod_{\hat{x}} \delta(Q_{\tilde{x}}, I) \tag{1.9}$$

where the summation variables $g_{x,\alpha}$ are defined on the edges of the lattice

$$Q_{\tilde{x}} = g_{x,1} g_{x+\hat{1},2} g_{x+\hat{2},1}^{-1} g_{x,2}^{-1} \tag{1.10}$$

and the δ-function is defined by the formula

$$\delta(g, I) = \begin{cases} 1 & \text{if} \quad g = I \\ 0 & \text{otherwise.} \end{cases}$$

In fact, the general solution of the connection equation $Q_{\tilde{x}} = I$ is

$$g_{x,\alpha} = g_x g_{x+\hat{\alpha}}^{-1}$$

and this brings us back to equation (1.5).

Systems on commutative groups are a special case, in which the δ-function in equation (1.9) can be factorized in the following way:

$$\delta(Q_{\tilde{x}}, I) = \sum_p \chi_p(Q_{\tilde{x}}) = \sum_p \chi_p(g_{x,1})\chi_p(g_{x+\hat{1},2})\chi_p^{-1}(g_{x+\hat{2},1})\chi_p^{-1}(g_{x,2}). \quad (1.11)$$

This sort of factorization is of decisive importance and allows for a unified presentation of the KW transform for all commutative groups.

We note that for a commutative group G all irreducible representations are one dimensional and their characters χ_p form a commutative group $\hat{G}$ (the character group) with a group multiplication defined in accordance with the tensor product of representations. By definition

$$\chi_{p_1 p_2}(g) = \chi_{p_1}(g)\chi_{p_2}(g) \qquad \chi_{p^{-1}}(g) = \chi_p^{-1}(g)$$

and the unit element of $\tilde{G}$ corresponds to the identity representation of G. Accordingly, the summation in equation (1.11) can be regarded as a summation over the elements of the dual group $\hat{G}$.

Substituting the expansion (1.11) in equation (1.9), an obvious regrouping of factors yields

$$\begin{aligned} Z &= \sum_{s_{x,\alpha}\in G} \prod_{x,\alpha} W(g_{x,\alpha}) \prod_{\tilde{x}} \sum_{p_{\tilde{x}}} \chi_{p_{\tilde{x}}}(g_{x,1})\chi_{p_{\tilde{x}}}(g_{x+\hat{1},2})\chi_{p_{\tilde{x}}^{-1}}(g_{x+\hat{2},1})\chi_{p_{\tilde{x}}^{-1}}(g_{x,2}) \\ &= \sum_{p_{\tilde{x}}\in G} \prod_{\tilde{x},\alpha} \tilde{W}\left(p_{\tilde{x}} p_{\tilde{x}+\hat{\alpha}}^{-1}\right) \end{aligned} \quad (1.12)$$

$$\tilde{W}\left(p_{\tilde{x}} p_{\tilde{x}+\hat{\alpha}}^{-1}\right) = \sum_{g\in G} W(g)\chi_{p_{\tilde{x}}}(g)\chi_{p_{\tilde{x}+\alpha}^{-1}}(g) = \sum_{g\in G} W(g)\chi_{p_{\tilde{x}} p_{\tilde{x}+\hat{\alpha}}^{-1}}(g). \quad (1.13)$$

The expression (1.12) defines a new, dual spin system on the dual group $\hat{G}$ with a new Hamiltonian $\tilde{H}$, which is defined by the formula

$$\exp\{-\tilde{H}(p)\} = \tilde{W}(p). \quad (1.14)$$

The result can be formulated in the following way.

Proposition 1.1. *A spin system on a commutative group G with a Hamiltonian $H(g)$ $(g \in G)$ is equivalent to a spin system on the character group $\hat{G}$ (and on the dual lattice) with the Hamiltonian $\tilde{H}(p)(p \in \hat{G})$ given by the Fourier transform*

$$\exp\{-\tilde{H}(p)\} = \sum_{g\in G} \exp\{-H(g)\}\chi_p(g). \quad (1.15)$$

This is a Kramers–Wannier transform. In contradistinction to the 'order variables' g_x the name 'disorder variables' can be given to the dual spins $p_{\tilde{x}}$.

2. Algebraic constructions[1]

(A) The group algebra $C(G)$ of G.

Let G be a finite group of order n with elements $\{g_1 = e, \ldots, g_n\}$.

[1] For further details of exploiting algebraic constructions one can consult the books [8, 9].

Definition 1. *The group algebra* $C(G)$ *of* G *is an* n*-dimensional algebra over the complex field* $\mathbf{C}$ *with basis* $\{g_1 = e, \dots g_n\}$. *A general element* $u = c(g) \in C(G)$ *is*

$$u = \sum \alpha_i g_i. \tag{2.1}$$

The product of two elements (convolution) $u, v \in C(G)$ *is defined as*

$$uv = \left(\sum_{i=1}^{n} \alpha_i g_i\right)\left(\sum_{i=1}^{n} \beta_j g_j\right) = \sum_{1}^{k} (\gamma_k g_k) \qquad \gamma_k = \sum_{g_i g_j = g_k} \alpha_i \beta_j. \tag{2.2}$$

(B) The ring of functions $C[G]$ on G.

Definition 2. $C[G]$ *is a linear space of all complex-valued functions on G and the product is defined pointwise:*

$$(f_1 \cdot f_2)(g) = f_1(g) f_2(g) \tag{2.3}$$

(C) let us determine the canonical pairing $\langle \cdot, \cdot \rangle$ of these two spaces

$$\begin{aligned} &C(G) \otimes C[G] \to \mathbf{C} \\ &\text{if} \quad u \in C(G) \quad \text{and} \quad f \in C[G] \quad \text{then} \\ &u \otimes f \to \langle u, f \rangle = \sum \alpha_i f(g_i). \end{aligned} \tag{2.4}$$

We choose as a basis in $C[G]$ functions such that $\langle g_i, g^j \rangle = \delta_i^j$ where δ_i^j is the Kronecker symbol.

This pairing enables us to identify $C(G)$ and $C[G]$ as vector spaces.

3. Canonical actions of group *G*

We now define two canonical representations, the adjoint representation on $C(G)$ and the regular representation on $C[G]$.

(A) $T(g) : C(G)$. The adjoint representation is defined on the basis consisting of elements of G by

$$g : g_i \to g g_i g^{-1}. \tag{3.1}$$

The adjoint representation $\mathrm{ad}\, G$ decomposes in the direct sum of irreducible representations and split $C(G)$ in the sum of subspaces invariant under the adjoint action.

Each irreducible subspace H_i relates to the orbit of $\mathrm{ad}\, G$ (3.1). The number of H_i is equal to m, the number of elements in the space $C(G)/[C(G), C(G)]$, where $[C(G), C(G)]$ denotes the commutant of $C(G)$.

(B) $\tilde{T}(g) : C[(G]$. Let us define the canonical representation $\tilde{T}$ in the space $C[G]$ as the (right) regular representation as

$$T(g) : f \Rightarrow T(g) : f(g_k) = f(g_k g) \qquad g \in G \qquad f(g) \in C[G]. \tag{3.2}$$

It is well known that in the decomposition of the regular representation into irreducible ones all irreducible representations appear with multiplicity equal to the dimension of the representation

$$\tilde{T} = \sum d_k V_k$$

where V_k is the irreducible representation of degree k and d_k is the degree (dimension) of V_k (multiplicity of irreducible representation).

Proposition 3.1. *The number m of irreducible representations $\tilde{T}$ is equal to the number of orbits of T.*

(C) The canonical scalar product in the space $C[G]$ is

$$\langle f_1, f_2\rangle = 1/n \sum_{k=1}^{n} f_1(g_k)\bar{f}_2(g_k) \qquad f_1, f_2 \in C[G]. \tag{3.4}$$

The characters $\chi_i(g)$ of the irreducible representation of G form the set of orthogonal functions with respect to the scalar product (3.4).

Now we construct the basis in the space $C[G]$. Let us choose the character $\chi_k(g)$ and act on $\chi_k(g)$ by the group G with the help of the right regular representation:

$$R_{g_l}\chi_k(g) \qquad l = 1, \ldots, n. \tag{3.5}$$

We obtain the space V_k with dim $V_k = |\chi_k(g)|^2$. As a result we get the factorization of $C[G]$:

$$C[G] = \sum_{k\in \mathrm{M}_G} V_k \qquad \mathfrak{M}_G = \{k = 1, \ldots, m_G\}$$

where m_G is the number of irreducible representations of G.

Orthonormalizing the set of functions (3.5) we obtain the basis in the space V_k. Since V_k are pairwise orthogonal, applying this procedure to all characters χ_k we obtain the desired basis in $C[G]$.

Definition 3. *We shall call the dual space $\hat{G}$ to G the basis in $C[G]$ which we construct in the section C.*

Motivations for such definition ensue from the case of a commutative group K. The characters of K are one dimensional and the action of G on characters is simply the multiplication on the scalar, the eigenvalue of the operator R_g. The derived basis is the same as the set of elements of the group $\hat{K}$.

4. The KW transform for finite groups

Let us consider the adjoint representation ad G of G, on the space $C(G)$, induced by

$$g : g_k \to g g_k g^{-1}.$$

Let us denote by g_k^G the orbit relative to the adjoint action for $g_k \in G$, and by $\delta_k \in C[G]$ its characteristic function

$$\delta_k(g_s) = \begin{cases} 1 & \text{if} \quad g_s \in g_k^G \\ 0 & \text{otherwise.} \end{cases}$$

Let m_G be the number of conjugacy classes relative to the adjoint action of G. Let us choose representations of the classes

$$g_1, \ldots, g_{k_j}.$$

Lemma 4.1. *A linear map*

$$W : C(G) \to \mathbf{C}$$

satisfies the condition

$$W(g_k) = W\big(g_l g_k g_l^{-1}\big) \tag{4.1}$$

for every $g_l \in G$,

$$W = \sum_{j=1}^{m} \gamma_j \delta_{k_j} \in C[G] = Hom(C(G), \mathbf{C})$$

i.e.

$$W(g_s) = \sum \gamma_j \delta_{k_j}(g_s). \tag{4.2}$$

We obtain a general form of the adjoint invariant linear mapping, if we choose as $\gamma = (\gamma_1, \ldots, \gamma_m)$, the vector of free parameters.

Now we shall find the form of a general linear mapping

$$\hat{W} : C[G] \to \mathbf{C}$$

determined by the characters $\chi^i(G)$.

The set of characters $\chi^1, \ldots, \chi^m$ of the irreducible representation of G form the orthonormalized basis (relative to the scalar product (3.4)) in $C[G]$. Here and further χ^1 is the character of the trivial one-dimensional representation.

We get $\hat{W} = \sum \hat{\gamma}_j \chi^j$ as

$$\hat{W}(\psi) = \sum_{j=1}^{m} \hat{\gamma}_j \langle \chi^j, \psi \rangle \tag{4.3}$$

since characters of representations by lemma 4.1 are ad-invariant functions, we introduce the matrix $\Gamma = \gamma_j^l$ using the expansion

$$\chi^l = \sum_{j=1}^{m} \gamma_j^l \delta_{k_j}. \tag{4.4}$$

Let us denote by $g^0, \ldots, g^{m-1}$ the orthonormalized basis in the algebra $C[G]$, dual to the basis $g_0, \ldots, g_{m-1}$ in the group algebra $C(G)$, i.e.

$$\langle g^i, g_j \rangle = \delta_j^i.$$

Let D be the duality map

$$D : C(G) \to C[G] \qquad D(g_k) = g^k. \tag{4.5}$$

Theorem 4.1. *If we pose*

$$\gamma_j = \sum_{l=1}^{m} \gamma_j^l \hat{\gamma}_l \qquad j = 1, \ldots, m$$

then by the canonical duality D the linear map

$$W : C(G) \to \mathbf{C} \qquad W(g) = \sum_{j=1}^{m} \gamma_j \delta_{k_j}(g)$$

passes to the linear map

$$\hat{W} : C[G] \to \mathbf{C} \qquad \hat{W}(\psi) = \sum_{j=1}^{m} \hat{\gamma}_j \langle \chi^j, \psi \rangle$$

and maps W and $\hat{W}$ themselves will be determined by the same function, more precisely

$$W(g_s) = \sum_{j=1}^{m} \gamma_j \delta_{k_j}(g_s) = n \sum_{j=1}^{m} \hat{\gamma}_j \chi^j(g_s) = n\hat{W}(g^s). \tag{4.6}$$

Proof. For any g_s we have

$$W(g_s) = \sum_{j=1}^{m} \gamma_j \delta_{k_j}(g_s) = \sum_{j=1}^{m}\sum_{l=1}^{m} \gamma_j^l \hat{\gamma}_l \delta_{k_j}(g_s) = \sum_{l=1}^{m} \hat{\gamma}_l \left(\sum_{j=1}^{m} \gamma_j^l \delta_{k_j} \right)(g_s)$$
$$= \sum_{l=1}^{m} \hat{\gamma}_l \chi^l(g_s) = n \sum_{l=1}^{m} \hat{\gamma}_l \langle \chi^l, g^s \rangle = n\hat{W}(g^s).$$

□

Definition 4. *We shall call the transform*

$$W(g_s) = \sum \gamma_j \delta_{k_j}(g_s) \to \hat{W}(g^s) = 1/n \sum_l \gamma_l \chi^l(g^s) \qquad \text{where} \quad \gamma_j = \sum_{l=1}^{m} \gamma_j^l \hat{\gamma}_l \tag{4.7}$$

is the Kramers–Wannier transform for finite groups.

In the next section we consider several examples which confirm the coincidence of our approach with the former one in the known cases and enables us to find explicit KW transforms in some earlier unknown cases.

5. Examples

(A) Commutative case $G = Z_n$. Let us consider first the special case $G = Z_3 = \{1, g, g^2\}$. In this case $\delta_j = \delta(g - g^{j-1})$, $j = 1, 2, 3$. Then

$$\begin{aligned} \chi^1 &= \delta_1 + \delta_2 + \delta_3 \\ \chi^2 &= \delta_1 + z\delta_2 + z^2\delta_3 \\ \chi^3 &= \delta_1 + z^2\delta_2 + z\delta_3 \qquad \text{as } z^4 = z \end{aligned} \tag{5.1}$$

where $z = \exp 2\pi \mathrm{i}/3$, and $\chi^k(g^j) = z^{(k-1)j}$ $(k = 1, 2, 3)$ are the characters of one-dimensional representations. Hence

$$\Gamma = \left(\gamma_j^l\right) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & z & z^2 \\ 1 & z^2 & z \end{pmatrix} \tag{5.2}$$

and we get $\hat{\gamma} = \Gamma^{-1}\gamma$.

If we choose $\gamma_1 = 1$, $\gamma_2 = \gamma_3 = \gamma$, we obtain

$$\hat{\gamma}_1 = \frac{1 + 2\gamma}{3} \qquad \hat{\gamma}_2 = \hat{\gamma}_3 = \frac{1 - \gamma}{3} \tag{5.3}$$

and hence

$$\hat{\gamma}_2/\hat{\gamma}_1 = \frac{1 - \gamma}{1 + 2\gamma}. \tag{5.4}$$

For the general case of the group Z_n we have to replace the formula (5.1) for characters $\chi^1, \ldots, \chi^n$ with

$$\begin{aligned}\chi^1 &= \delta_1 + \delta_2 + \cdots + \delta_n \\ \chi^2 &= \delta_1 + z\delta_2 + z^{n-1}\delta_n \\ &\vdots \\ \chi_l^n &= \delta_1 + z^{(n-1)}\delta_2 + z\delta_n\end{aligned} \tag{5.5}$$

and for $\Gamma = \left(\gamma_j^l\right)$ we get

$$\Gamma = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & z & \cdots & z^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & z^{n-1} & \cdots & z \end{pmatrix} \tag{5.6}$$

In the special case of choosing parameters γ_j:

$$\gamma_1 = 1, \gamma_2, \ldots, \gamma_n = \gamma$$

we obtain

$$\frac{\hat{\gamma}_j}{\hat{\gamma}_1} = \frac{1-\gamma}{1+(n-1)\gamma}. \tag{5.7}$$

These formulae coincide with the similar one in [2].

(B) The group S_3. This is the first non-trivial example of non-Abelian groups which was studied in [2]. Following our general approach we split the group S_3 in three classes of conjugacy elements or three orbits:

$$S_3 = \{\Omega_1 = \{e\}\ \Omega_2 = \{a, a^2\}\ \Omega_3 = \{b, ab, a^2b\}\}.$$

The characteristic functions are

$$\delta_1 = \delta(\Omega_1) = \delta(g-e) \qquad \delta_2 = \delta(\Omega_2) \qquad \delta_3 = \delta(\Omega_3).$$

Following our general procedure (see (4.4)) and using

$$\chi^1 = \delta_1 + \delta_2 + \delta_3 \qquad \chi^2 = \delta_1 + \delta_2 - \delta_3 \qquad \chi^3 = 2\delta_1 - \delta_2$$

we get the matrix

$$\Gamma = \left(\gamma_j^l\right) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

and hence $\hat{\gamma} = \Gamma^{-1}\gamma$

$$\begin{aligned}\hat{\gamma}_1 &= \tfrac{1}{6}(\gamma_1 + 2\gamma_2 + 3\gamma_3) \\ \hat{\gamma}_2 &= \tfrac{1}{6}(\gamma_1 + 2\gamma_2 - 3\gamma_3) \\ \hat{\gamma}_3 &= \tfrac{1}{3}(\gamma_1 - \gamma_2)\end{aligned} \tag{5.8}$$

with the following relation:

$$\hat{\gamma}_1 + \hat{\gamma}_2 + 2\hat{\gamma}_3 = \gamma_1.$$

If we choose the free parameters $\gamma_1, \gamma_2, \gamma_3$ as $1, \gamma_2, \gamma_3$ we obtain two independent parameters $\hat{\eta}_1, \hat{\eta}_2$

$$\hat{\eta}_1 = \frac{\hat{\gamma}_2}{\hat{\gamma}_1} = \frac{1 + 2\gamma_2 - 3\gamma_3}{1 + 2\gamma_2 + 3\gamma_3} \qquad \hat{\eta}_2 = \frac{\hat{\gamma}_3}{\hat{\gamma}_1} = \frac{2(1-\gamma_2)}{1 + 2\gamma_2 + 3\gamma_3} \tag{5.9}$$

which coincide with the formula (5.7) in [2].

Remark 1. Let us mention the missing factor 2 in the nominator of $\hat{\eta}_2$ in (5.7) in [2].

(C) The group of icosahedron I_5. This group is isomorphic to A_5, the group of even permutations of five elements. The order of the group is equal to 60. The theory of irreducible representations of A_5 is well known, see e.g. [6]. We summarize the necessary facts. There are five classes of conjugacy with representatives

$$e, a_1 = (12)(34) \qquad a_2 = (123) \qquad a_3 = (12345) \qquad a_4 = (21345). \tag{5.10}$$

The number of characters of irreducible representations is also equal to 5. Following [6] we present the table of characters:

	e	(12)(34)	(123)	(12345)	(21345)
χ^1	1	1	1	1	1
χ^2	3	−1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ^3	3	−1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ^4	4	0	1	−1	−1
χ^5	5	1	−1	0	0
	1	15	20	12	12

The last line indicates the number of elements in each conjugacy class. The characteristic functions have the form

$$\delta_1 = \delta(\Omega_1) = \delta(g-e) \qquad \delta_2 = \delta(\Omega_2) \qquad \delta_3 = \delta(\Omega_3) \qquad \delta_4 = \delta(\Omega_4) \qquad \delta_5 = \delta(\Omega_5)$$

where Ω_i ($i = 1, 2, 3, 4, 5$) are orbits corresponding to conjugacy classes in (5.10).

Applying our general procedure (see (4.4)) and using

$$\begin{aligned}
\chi^1 &= \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 \\
\chi^2 &= 3\delta_1 - \delta_2 + \varepsilon\delta_4 + \eta\delta_5 \\
\chi^3 &= 3\delta_1 - \delta_2 + \eta\delta_4 + \varepsilon\delta_5 \\
\chi^4 &= 4\delta_1 + \delta_3 - \delta_4 - \delta_5 \\
\chi^5 &= 5\delta_1 + \delta_2 - \delta_3
\end{aligned}$$

we denote $\frac{1+\sqrt{5}}{2}$ by ε and $\frac{1-\sqrt{5}}{2}$ by η.

We obtain the (transpose) matrix of characters Γ' having the form

$$\begin{pmatrix} 1 & 3 & 3 & 4 & 5 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & \varepsilon & \eta & -1 & 0 \\ 1 & \eta & \varepsilon & -1 & 0 \end{pmatrix}$$

and we get the inverse matrix Γ'^{-1}

$$\begin{pmatrix} \frac{1}{60} & \frac{1}{4} & \frac{1}{3} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{20} & -\frac{1}{4} & 0 & \frac{\varepsilon}{5} & \frac{\eta}{5} \\ \frac{1}{20} & -\frac{1}{4} & 0 & \frac{\eta}{5} & \frac{\varepsilon}{5} \\ \frac{1}{15} & 0 & \frac{1}{3} & -\frac{1}{5} & -\frac{1}{5} \\ \frac{1}{12} & \frac{1}{4} & -\frac{1}{3} & 0 & 0 \end{pmatrix} \tag{5.11}$$

Hence $\hat{\gamma} = \Gamma'^{-1}\gamma$:

$$\begin{aligned}
&\frac{1}{60}\gamma_1 + \frac{1}{4}\gamma_2 + \frac{1}{3}\gamma_3 + \frac{1}{5}\gamma_4 + \frac{1}{5}\gamma_5 = \hat{\gamma}_1\\
&\frac{1}{20}\gamma_1 - \frac{1}{4}\gamma_2 + \frac{\sqrt{5}+1}{10}\gamma_4 + \frac{1-\sqrt{5}}{10}\gamma_5 = \hat{\gamma}_2\\
&\frac{1}{20}\gamma_1 - \frac{1}{4}\gamma_2 + \frac{1-\sqrt{5}}{10}\gamma_4 + \frac{1+\sqrt{5}}{10}\gamma_5 = \hat{\gamma}_3\\
&\frac{1}{15}\gamma_1 - \frac{1}{3}\gamma_3 - \frac{1}{5}\gamma_4 - \frac{1}{5}\gamma_5 = \hat{\gamma}_4\\
&\frac{1}{12}\gamma_1 - \frac{1}{4}\gamma_2 - \frac{1}{3}\gamma_3 = \hat{\gamma}_5.
\end{aligned} \tag{5.12}$$

(D) Dihedral group D_n. The group D_n is the group of symmetry of the regular polygon M_n with n sides. The order of the group D_n is $2n$. D_n includes the group C_n of rotation of the polygon M_n which is the cyclic of order n. Let b be some reflection of M_n, the group D_n then generates the elements $a \in C_n$, $a^n = 1$ and b, $b^2 = 1$ with the relation

$$bab^{-1} = a^{-1}.$$

So each element of D_n can be represented in the form

$$ba^k \qquad 0 \leqslant k \leqslant n-1$$

The group of characters of D_n is well-known, see e.g. [6].

We consider two different cases

1. $D_n(n - even)$. These are the four characters of one-dimensional representations of D_n

	e	a^k	ba^k
χ^1	1	1	1
χ^2	1	1	-1
χ^3	1	$(-1)^k$	$(-1)^k$
χ^4	1	$(-1)^k$	$(-1)^{k+1}$.

Two-dimensional representations ρ^h have the form

$$\rho^h(a^k) = \begin{pmatrix} z^{hk} & 0 \\ 0 & z^{-hk} \end{pmatrix} \qquad \rho^h(ba^k) = \begin{pmatrix} 0 & z^{-hk} \\ z^{hk} & 0 \end{pmatrix} \tag{5.13}$$

and the corresponding characters

$$\chi_h(a^k) = z^{hk} + z^{-hk} = 2\cos\pi h/k \qquad z = \exp(2\pi\mathrm{i}/n) \qquad \chi_h(ba^k) = 0. \tag{5.14}$$

We get $(\frac{n}{2} - 1)$ two-dimensional irreducible representations.

2. $D_n(n - odd)$. In this case we have only two one-dimensional representations with the characters

	a^k	ba^k
χ^1	1	1
χ^2	1	-1

The two-dimensional representations and characters are determined by the same formula as in (5.13) and (5.14).

In this case we get $(n-1)/2$ two-dimensional irreducible representations. Following the classical theorem of Frobenius, that the sum of the square of dimensions of all the irreducible representations is equal to the order of the group, we conclude that we obtain all irreducible representations of the dihedral group D_n. Using the result of section 4 we construct the KW transform for the general D_n. It is a cumbersome but straightforward procedure. Since we already considered the special case $S_3 = D_3$, we choose the simple example of the group D_n with even index.

The simplest non-trivial case is D_4.

Example D_4. The group D_4 has two generators a, b with the relations $a^4 = b^2 = e$ and $bab^{-1} = a^{-1}$.

There are five classes of conjugacy

$$e, a^2, \{a, a^3\}, \{b, a^2b\}, \{ab, a^3b\}\}$$

the table of characters is

	e	a^2	$\{a; a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
χ^1	1	1	1	1	1
χ^2	1	1	1	-1	-1
χ^3	1	1	-1	1	-1
χ^4	1	1	-1	-1	1
χ^5	2	-2	0	0	0

We have five characteristic functions δ_i of orbits

$$\Omega_1 = e \qquad \Omega_2 = a^2 \qquad \Omega_3 = \{a, a^3\} \qquad \Omega_4 = \{b, a^2b\} \qquad \Omega_5 = \{ab, a^3b\}$$

and five characteristic functions

$$\delta_1(e), \delta_2(\Omega_2), \delta_3(\Omega_3), \delta_4(\Omega_4), \delta_5(\Omega_5).$$

In our case equations (5.1) are the following:

$$\begin{aligned}
\chi^1 &= \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5\\
\chi^2 &= \delta_1 + \delta_2 + \delta_3 - \delta_4 - \delta_5\\
\chi^3 &= \delta_1 + \delta_2 - \delta_3 + \delta_4 - \delta_5\\
\chi^4 &= \delta_1 + \delta_2 - \delta_3 - \delta_4 + \delta_5\\
\chi^5 &= 2\delta_1 - 2\delta_2.
\end{aligned}$$

Matrix $\Gamma = \left(\gamma_j^l\right)$ has the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & -1 & -1\\ 1 & 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1 & 1\\ 2 & -2 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\Gamma^{-1} = 1/8 \begin{pmatrix} 1 & 1 & 2 & 2 & 2\\ 1 & 1 & 2 & -2 & -2\\ 1 & 1 & -2 & 2 & -2\\ 1 & 1 & -2 & -2 & 2\\ 2 & -2 & 0 & 0 & 0 \end{pmatrix}$$

Hence $\hat{\gamma} = \Gamma^{-1}\gamma$ and

$$\begin{aligned}
&\tfrac{1}{8}\gamma_1 + \tfrac{1}{8}\gamma_2 + \tfrac{1}{4}\gamma_3 + \tfrac{1}{4}\gamma_4 + \tfrac{1}{4}\gamma_5 = \hat{\gamma}_1 \\
&\tfrac{1}{8}\gamma_1 + \tfrac{1}{8}\gamma_2 + \tfrac{1}{4}\gamma_3 - \tfrac{1}{4}\gamma_4 - \tfrac{1}{4}\gamma_5 = \hat{\gamma}_2 \\
&\tfrac{1}{8}\gamma_1 + \tfrac{1}{8}\gamma_2 - \tfrac{1}{4}\gamma_3 + \tfrac{1}{4}\gamma_4 - \tfrac{1}{4}\gamma_5 = \hat{\gamma}_3 \\
&\tfrac{1}{8}\gamma_1 + \tfrac{1}{8}\gamma_2 - \tfrac{1}{4}\gamma_3 - \tfrac{1}{4}\gamma_4 + \tfrac{1}{4}\gamma_5 = \hat{\gamma}_4 \\
&\tfrac{1}{4}\gamma_1 - \tfrac{1}{4}\gamma_2 = \hat{\gamma}_5
\end{aligned}$$

Remark 2. It is well known that group D_4 has the same character tables as the group of quaternions Q. Hence, the set of characters is insufficient to distinct spin systems with values in groups D_4 and Q. However, these systems have the different structure of dual spaces $\hat{D}_n$ and $\hat{Q}$ in the sense of definition 3.

This follows, for example, that the group D_4 has a cyclic subgroup of order 4 and the group Q has not.

There also exists another characteristic, the so-called generalized characters introduced by Frobenius [10]. Generalized characters of order k or k-characters are mappings from $\underbrace{G \cdot G \cdot \ldots G}_{k} \to \mathbf{C}$ which for $k = 1$ are ordinary characters, see details in [11]. The remarkable property of k-characters is the following. One, two and three characters completely determined the group G. In the case of groups D_4 and Q the non-trivial 2-characters are distinct.

6. Conclusion

Our approach to the KW transform has important applications. We briefly discuss some of them, intending to return to these problems in forthcoming publications.

(A) KW transforms and compact groups. All the constructions of sections 3 and 4 remain valid if we replace finiteness of the group G with compactness. It means that KW transforms can be constructed for spin systems with compact group of symmetry.

(B) KW transforms and quantum groups. We refer the reader to [12, 13] for all notation and following references in the theory of Hopf algebras and quantum groups.

Let us consider the algebra $C[G]$. If we endow $C[G]$ by the operation of coproduct $\triangle : C[G] \to C[G] \otimes C[G]$ induced by the multiplication in the group G, the algebra $C[G]$ becomes a Hopf algebra. Using the natural dual to $C[G]$, the algebra $C(G)$, we are able to construct another Hopf algebra, (quantum) double $D(G) = C[G] \otimes C(G)$ [12]. Since transformations W and $\hat{W}$ act as $W : C(G) \to \mathbf{C}$ and $\hat{W} : C[G] \to \mathbf{C}$, i.e. $W \in C[G] = Hom(C(G), \mathbf{C})$ and $\hat{W} \in C[G] = Hom(C(G), \mathbf{C})$ that is $W \otimes \hat{W} \in D(G)$. The KW transform yields explicit solutions of Yang–Baxter equations related to the quantum group $D(G)$.

This observation leads to very explicit formulae in the structure theory of quantum groups and quantum spin systems.

And last but not least,

(C) In our paper we consider spin systems with a global non-Abelian symmetry. It is natural to ask about generalizing the proposed technique to systems with a local (gauge) symmetry. The study of such systems including Ising and Potts chiral models, Abelian and non-Abelian gauge fields is very important for quantum field theory and the theory of phase transitions.

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